Paper II (B.Sc Part I) Hons.

Exact Differential.

Definition: — If an expression of the type $Mdx + Ndy$ (M and N are functions of x and y or constant) may be reduced to du, where u is a function of x and y, then $Mdx + Ndy$ is said to be an exact differential.

Theorem: The necessary and sufficient condition that the expression $Mdx + Ndy$ be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof: — (i) The condition is necessary.

Let the given expression $Mdx + Ndy$ be an exact differential and let it be equal to du.

Now $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

i.e. $Mdx + Ndy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

on comparison, we get

$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \cdot \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Since $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) The condition is sufficient.

We have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and we are to prove that $Mdx + Ndy = du$.

Let $\int M dx = P$ $\therefore M = \partial P/\partial x$

Now $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$

$\Rightarrow N = \frac{\partial P}{\partial y} + \phi(y)$ where $\phi(y)$ is a function of y,

therefore we have

$$M dx + N dy = \frac{\partial P}{\partial x} . dx + \frac{\partial P}{\partial y} dy + \phi(y) dy$$

$$= d[P + \psi(y)] \text{ where } d(\psi(y)) = \phi(y) dy$$

Now let $P + \psi(y) = u$ where u is a function of x and y

$$\therefore M dx + N dy = du.$$

Hence the result.

Example:– Prove that the expression $(x^3 + 3xy^2)dx + (y^3 + 3x^2y) dy$ is an exact differential.

Solution:– Here $M = x^3 + 3xy^2$

and $N = y^3 + 3x^2y$

$$\therefore \frac{\partial M}{\partial y} = 3x \cdot 2y = 6xy$$

and $\frac{\partial N}{\partial x} = 3.2.x.y = 6xy$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Proved.

Anjani Kumar Singh.